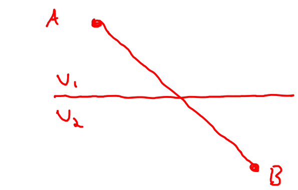
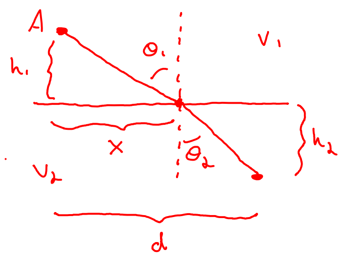
Let's start w/ Fermat (~1600) and his observation that light tends to travel in straight lines, but "bends" at an interface. He observed that the bending could be explained if light follows the path of least time.



Path of shortest length (also shortest time if $v_1 = v_2$)



Given h_1, h_2, v_1, v_2, d find x s.t. t_{tot} is minimum.

$$t_1 = \frac{\sqrt{h_1^2 + x^2}}{v_1} \qquad t_2 = \frac{\sqrt{h_2^2 + (d-x)^2}}{v_2}$$

$$t_{tot} = t_1 + t_2$$

$$\frac{dt_{tot}}{dx} = \frac{x}{v_1\sqrt{h_1^2+x^2}} - \frac{(d-x)}{v_2\sqrt{h_2^2+(d-x)^2}} = 0$$

But $\sin\theta_1 = \frac{x}{\sqrt{h_1^2+x^2}} \qquad \sin\theta_2 = \frac{(d-x)}{\sqrt{h_2^2+(d-x)^2}}$

Then: $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$

Friday, January 26, 2018
9:40 AM

Now let's see if we can "relativize" Fermat's principle. First of all, which "time" should we extremize?
The only distinguished time in relativity is the proper time which we defined as:

$$\tau = \int_A^B \sqrt{-ds^2} \Rightarrow \text{in 2D flat spacetime } \eta_{\mu\nu} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \text{ w/ } (ct, x) \text{ this becomes:}$$

$$\tau = \int_A^B \sqrt{c^2 dt^2 - dx^2} = \int_A^B \sqrt{c^2 - v^2}\, dt = \int_A^B f(v)\, dt$$

We want the path $x(t)$ which extremizes τ, so we set $\delta\tau = 0$ under variations which vanish @ A, B.

$$\delta\tau = \int_A^B \delta f(v)\, dt = \int_A^B \frac{\partial f}{\partial v} \delta v\, dt = \int_A^B \frac{\partial f}{\partial v} \delta \frac{dx}{dt}\, dt = \int_A^B \underbrace{\frac{\partial f}{\partial v} \frac{d\,\delta x}{dt}}\, dt$$

Integrate by parts using: $\frac{d}{dt}\left[\frac{\partial f}{\partial v} \delta x\right] = \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right)\delta x + \overbrace{\frac{\partial f}{\partial v}\frac{d\delta x}{dt}}$ then:

$$\delta\tau = \underbrace{\int_A^B \frac{d}{dt}\left[\frac{\partial f}{\partial v} \delta x\right] dt}_{\frac{\partial f}{\partial v} \delta x \big|_A^B} - \int_A^B \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right) \delta x\, dt = 0$$

but $\delta x(t_A) = \delta x(t_B) = 0$.

$$\delta\tau = -\int_A^B \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right)\delta x\, dt = 0$$

We want this true for arbitrary δ $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial v}\right) = 0$$

$$\frac{d}{dt}\left(\frac{-v}{\sqrt{c^2 - v^2}}\right) = 0$$

$$\underbrace{c^2\gamma^3}_{\neq 0}\dot{v} = 0$$

$$\Rightarrow \dot{v} = 0$$

Which is just a constant velocity path from A to B.

What if we are not in flat spacetime? Or use coordinates other than (ct, x, y, z)?

$$\tau = \int_A^B \sqrt{-g_{\mu\nu}\,dx^\mu dx^\nu}$$

Now since there are no preferred coordinates we really should be parameterizing the path w/ τ (not t).

$$\tau = \int_A^B \sqrt{-\underbrace{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}_{f}}\;d\tau = \int_A^B \sqrt{-f}\,d\tau$$

We want:

$$\delta\tau = \int_A^B \delta\sqrt{-f}\,d\tau = -\int_A^B \frac{\delta f}{2\sqrt{-f}}\,d\tau = 0$$

But recall: $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = U^\mu U_\nu = -1$ so then:

$$\delta\tau = -\frac{1}{2}\int_A^B \delta f\,d\tau = 0$$

but this implies that we are also extremizing $I = \frac{1}{2}\int_A^B f\,d\tau$ which is much easier to handle than the form with $\sqrt{}$

Consider varying: $x^\mu \to x^\mu + \delta x^\mu$

$g_{\mu\nu} \to g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda$ (Since the metric varies over spacetime, i.e. depends on x^λ, different paths will experience different forms of the metric)

Then:

$$\delta I = \frac{1}{2}\int_A^B \delta\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)d\tau = \frac{1}{2}\int_A^B \left[\partial_\lambda g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\lambda + \underbrace{g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}}\right]d\tau$$

We can integrate the last two terms using:

$$\frac{d}{d\tau}\left[g_{\mu\nu}\,\delta x^\mu \frac{dx^\nu}{d\tau}\right] = \frac{\partial g_{\mu\nu}}{d\tau}\delta x^\mu\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\,\delta x^\mu\frac{d^2x^\nu}{d\tau^2}$$

So:

$$\int_A^B g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau = \underbrace{\int_A^B \frac{d}{d\tau}\left[g_{\mu\nu}\,\delta x^\mu\frac{dx^\nu}{d\tau}\right]d\tau}_{=0 \text{ for } \delta x^\mu(\tau_A) = \delta x^\mu(\tau_B) = 0} - \int_A^B\left[\frac{\partial g_{\mu\nu}}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\mu\,d\tau$$

Altogether we have:

$$\delta I = \frac{1}{2}\int_A^B\left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\lambda - \underbrace{\underbrace{\frac{\partial g_{\mu\nu}}{d\tau}}_{\frac{\partial g_{\mu\nu}}{dx^\alpha}\frac{dx^\alpha}{d\tau}}\frac{dx^\nu}{d\tau}\delta x^\mu}_{\mu\to\lambda,\ \alpha\to\mu} - \underbrace{g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\delta x^\mu}_{\mu\to\lambda} - \underbrace{\underbrace{\frac{\partial g_{\mu\nu}}{d\tau}}_{\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}}\frac{dx^\mu}{d\tau}\delta x^\nu}_{\nu\to\lambda,\ \alpha\to\nu} - \underbrace{g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\delta x^\nu}_{\nu\to\lambda,\ \mu\to\nu}\right]d\tau$$

$$\delta I = \frac{1}{2}\int_A^B\left[\partial_\lambda g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \partial_\mu g_{\lambda\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2} - \partial_\nu g_{\mu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - g_{\nu\lambda}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\lambda\,d\tau$$

But for arbitrary δx^λ this requires: $-2g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2} + (\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$

Applying $\frac{-1}{2}g^{\beta\lambda}$ to both sides: $\frac{d^2x^\beta}{d\tau^2} - \frac{1}{2}g^{\beta\lambda}(\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$

recall $g^{\beta\lambda}g_{\lambda\nu} = \delta^\beta_\nu$

or using: $\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$

we have: $\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ The geodesic equation!

Friday, January 26, 2018
11:52 AM

What about good old $S = \int L\,dt$ w/ $L = T - U$?

Well recall that for SR we extremized $\tau = \int_A^B \sqrt{c^2 - v^2}\,dt = \int_A^B c\sqrt{1 - \frac{v^2}{c^2}}\,dt$

For $\frac{v}{c} \ll 1$ this is: $\tau \simeq \int_A^B c\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \ldots\right)dt = \text{constant} + \frac{1}{c}\underbrace{\int_A^B \frac{1}{2}v^2\,dt}_{\int T\,dt}$